CHAOS

Verhulst’ population model
Verhulst applied his logistic equation for population growth to demographic studies. The model was cast to produce solutions for a single population that grows from a small initial number of individuals to a limited population with an upper bound implicitly set by factors such as food supply, living space or pollution.

The concept embodied in the model was one of a natural decline in the rate of population growth with time as population increases in size. The solutions of the logistic oppose the effect of an unspecified limiting factor depending on population density against exponential Malthusian growth.

The model’s inferences had reasonable correspondence to, and therefore confirmation in, the more rapid population growth in eighteenth century America as compared to the slower rate that existed in Europe.

The logistic equation states that the rate of change of the population \( \frac{dN}{dt} \) with time \( t \) depends in some way on population, that is, a function of the population, \( f(N) \).

Derivation of a logistic equation can begin with the differential relation implied above or with a finite approach. In either case, general analytic expressions are made to fit population realities by mathematical constraints.
With the differential equation approach, a general expression of how populations change is:

\[
dN/dt = f(N)
\]

requiring integration for a solution that predicts \(N\) as a matter of time. There is an interesting way to express \(F(N)\) that will allow a step-wise examination of what it means. The function is replaced by a sequence of additive terms known as a polynomial, a technique that was developed by another mathematician, Brook Taylor [1685-1731]. The polynomial is known as the Taylor series:

\[
f(N) = a + bN + cN^2 + dN^3 + \ldots
\]

and so on, where the initial alphabet letters in each term stands for a constant that allows the series to approximate \(f(N)\), whatever is \(f(N)\). The more terms used, the better the approximation. The step-wise development begins here by taking the approximation of \(f(N)\) one more term for each integration. But first, we assume parenthood is part of the deal, so when there is no population, that is \(N=0\), things just aren’t going anywhere, \(dN/dt = 0\) and so the first constant is 0, that is, \(a = 0\). So

\[
f(N) = bN + cN^2 + dN^3 + \ldots
\]

We do the calculus just twice, taking only the first term in the integration; and then two terms: the first model is given by the differential equation \(dN/dt = bN\) and by intergrating, \(N(t) = e^{bt}\), where \(b\) is interpreted as (birth rate - death rate). This means the population can grow exponentially in time without bounds. The prediction is often associated with Robert Malthus’ contention that demand for food will become much greater than the supply. This because population will increase geometrically while food grows at a slower, arithmetic rate. Eventually, some disaster checks the population.
Taking the linear and quadratic terms of the Taylor series generates the next level of complexity:

\[
dN/dt = bN + cN^2 = bN(K-N)/K
\]

The constant \( c \) is evaluated by recognizing that \( dN/dT = 0 \) when the population has reached its upper limit, \( K \): \( c = -b/K \). Then the integration yields

\[
N(t) = K / (1 + e^{-bt})
\]

which is a version of Verhulst’s logistic model. Below are graphical representations of the growth of a population (on the left) and the population’s rate of change plotted against time for a logistic. The population curve has been given the name \textit{sigmoid}. The highest rate of population change, the peak in the rate of change occurs at the time of the inflection in the sigmoid curve. Curves such as these were used by Verhulst in his studies of population growth and demography.
The finite approach to formulating a Verhulst relation can begin with a simple expression of how a given population grows; in words this can be:

the 2004 population will be twice as large as 2003.

\[ P_{2004} = 2 \times P_{2003} \]

If we get just a little clever and let the indexing number \( j \) stand sequentially for generations, then

\[ P_{j+1} = 2 \times P_j \]

And to allow the growth to be more or less double

\[ P_{j+1} = C \times P_j \]

where \( C \) is the population birthrate, a constant, while the population changes generations on and on. Note that the equation represents the fact that next the generation’s number of individuals is directly proportional to the number in the preceding generation (no real surprises here). Now to add a new term that will account for control of the population growth, in an implied sense. Environmental factors such as limited space or lack of food (remember Mathus’ war and famine disasters) put on a population’s brakes).

First, we now take the population (\( P \)) to be a percentage of its maximum attainable size (whatever that is) so the value of \( P \) will range from zero to one. More cleverness: we need a term that will make the population’s rate of change go to zero when the population approaches maximum. Use \((1 - P_j)\) as a multiplying factor to make this happen:

\[ P_{j+1} = C \times (1 - P_j) \times P_j \]

It works like this: when \( P_j \) is small, the equation is very nearly like \( P_{j+1} = C \times P_j \) since \((1 - P_j)\) has a value very near one and so the population will increase in a Malthusian sense with very little bounds. When \( P_j \) is relatively large (remember its range is zero to one) then the new term becomes very small and, relatively, next generation’s population diminishes. The equation can now be put in the form

\[ P_{j+1} = C \times P_j - C \times P_j^2 \]
The population model has now expanded to include the influence of the term $CP_j^2$. This additional term is quadratic and the equation is what is known as non-linear, that is, there is no longer a direct one-to-one relationship between the progression of time (or generations) and the associated size of the population. Subtle effects are introduced in the population predictions. Large generation-to-generation shifts in population exist within the realm of the model’s solutions along with sigmoid, exponential decay, and periodic solutions. The non-linear effect are possible through the quadratic term but what triggers the chaotic behavior is the value of the birthrate, $C$.

Different birth rates can bring on very different populations when comparisons are made after many generations. The primary interest is in population stability and size, that is, at when and what number of individuals is the population going to find an equilibrium. When the birthrate is less than $C = 1$, the population does not grow but rather dies off. From about $C = 1.2$ to about 2.7, the population tends to stabilize and approach a definite limiting value.

When a value of $C = 3$ is reached, solutions to the logistic equation begin to exhibit startling results. At first, the limits oscillate between two distinct values, that is, there are two possible asymptotic population values that alternate generation after generation. Beyond $C = 3.4$, the limits continue to oscillate, but now between four values.

What is happening is made clear when a graph of limiting population values is plotted against associated birthrates for a wide range of $C$, say from essentially zero to four.
When $C = 3.7$, the nature of the limits change drastically with little repetition, as well, the solution does not exhibit much in the way of stability until generations sixty to seventy and that is transient since the oscillations in the population begin once again above a value of 70.
One can perform a step-wise examination of the logistic to a parametric variation in birthrate and accumulate, as data, long-term asymptotic population levels. Birthrate and final population size can be plotted against one another to reveal the fact that a given birth rate can generate more than one possible final population. In fact, the solution at particular values of birth rate splits in two, that is, bifurcates. The data below shows the results of such a parametric study of the logistic.

<table>
<thead>
<tr>
<th>#</th>
<th>Birthrate (C)</th>
<th>Final Population Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>1.50</td>
<td>0.33</td>
</tr>
<tr>
<td>3</td>
<td>2.00</td>
<td>0.50</td>
</tr>
<tr>
<td>4</td>
<td>2.50</td>
<td>0.60</td>
</tr>
<tr>
<td>5</td>
<td>2.75</td>
<td>0.64</td>
</tr>
<tr>
<td>6</td>
<td>2.95</td>
<td>0.66</td>
</tr>
<tr>
<td>7</td>
<td>3.00</td>
<td>0.70 0.63</td>
</tr>
<tr>
<td>8</td>
<td>3.10</td>
<td>0.76 0.56</td>
</tr>
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<td>9</td>
<td>3.20</td>
<td>0.80 0.51</td>
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<tr>
<td>11</td>
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<td>0.84 0.45</td>
</tr>
<tr>
<td>12</td>
<td>3.45</td>
<td>0.86 0.84 0.42 0.46</td>
</tr>
<tr>
<td>13</td>
<td>3.50</td>
<td>0.87 0.83 0.50 0.38</td>
</tr>
</tbody>
</table>

As birthrate increases, the number of bifurcations increase, branch to branch. The data graph above illustrate bifurcations at birthrates 3 and 3.45.
The bifurcations continue to proliferate (below, left) as a parametric treatment of birthrate is carried to higher values. However, the intervals between bifurcations decrease until very small changes in birthrate produce many possible “end states for the logistic system”, that is, many possible final population values for a given birthrate. The result of carrying the logistic equation to values above four produces a chaotic graph in which virtually any final prediction value from 0 to 1 occurs (above). Recall that population, as we defined it, is a percentage of a population’s maximum size, K.
Verhulst used the logistic equation to model human populations. Other circumstances and situations can be investigated within the analytic framework of the logistic: the performance of an amplifier with a negative feedback circuit and the influence of interest rates on financial markets.

By virtue of its quadratic nature, the logistic equation can produce population predictions that do not, in the long term, approach stable final values. Why wasn’t this examined? The answer lies in human fecundity and birthrates that produce model solutions that match actual census data. We reproduce with no where near the ferocity of some other species, especially insects, and so human numbers, not counting catastrophes like Black Plague, influenza, and HIV, are relatively stable and do not fluctuate as do year-to-year mosquito infestations.

However, indeterminacy exists with Verhulst’ models, they are, after all, non-linear (quadratic) in nature. The “route to chaos” that is followed through the model solutions is known as period doubling, one of several ways chaos can develop.

When the birthrate remains between values of 1 to 3, the long term population approaches a stable, fixed size. The possible final states of the population doubles at a value of 3 and requires two generations for repetition to occur. This is evident in the data on slide 8 and the accompanying graph. At a value of 3.45, the number of possible final population values doubles again to 4 and requires 4 generations for repetition (slide 8). The doubling continues, occurring with smaller and smaller birthrate increments: 3.54 (8), 3.564 (16), 3.569 (32) … then, at $C = 3.57$, clearly recognizable disorganization ensues and the long-term population values never approach fixed values. However, for birthrate values between 3.57 and 4, there are regions of chaotic solutions interspersed with stable periodic ones (slide 9). This is yet another characteristic of chaotic domains.
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